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-USSR-

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Introduction

The necessity for realizing the optimal process on any given criterion in an automatic control system with limited control functions (actuating signals) has lead to the examination of a certain problem in the calculus of variations. Such a problem in general cannot be solved by means of the well-known Euler equations since the control function varies only within a bounded and closed domain.

The present paper contains a proof of L.S. Pontryagin's Maximum Principle for linear control systems which is used as a basis in determining the optimal control function when it is limited by a closed domain.

For the sake of simplicity, the article will deal only with optimum-rate systems; in general, however, the Maximum Principle is equally applicable to other criteria in the form of an integral of functions of the system coordinates and controlling functions.

1. Statement of the Problem

We now proceed to state precisely the problem of optimal control with linear controlling function. In this case, the Maximum Principle is proved with the aid of classi-

cal infinitesimal variations of the control functions. Thus, let there be a differential equation of order n which describes the transient process in a given automatic control system:

$$x^{(n)} = f(x, x', x'', \ldots, x^{(n-1)}) + u(t). \tag{1}$$

Introducing the notation

$$x = x^1,\ x' = x^2, \ldots, x^{(n-1)} = x^n,$$

it is possible to rewrite equation (1) as a system of n-th order differential equations:

$$\begin{aligned} \dot{x}^1 &= x^2; \\ \dot{x}^2 &= x^3; \\ &\ldots\ldots\ldots\ldots \\ \dot{x}^{n-1} &= x^n. \\ \dot{x}^n &= f(x^1, x^2, \ldots, x^n) + u(t), \end{aligned} \tag{2}$$

where u(t) is a scalar control vector.

Let us assume that $u = u(t)$ is a piecewise continuous function of time t, $|u(t)| \leq 1$ for any instant of time t. The last statement is a condition constraining the control function to lie within the limits of variation of u. The control $u = u(t)$ with such properties will be termed a permissible control.

For the sake of brevity and symmetry in notation, let us replace system (2) by a more general system of n-th order differential equations

$$\dot{x}^i = f^i(x^1, x^2, \ldots, x^n) + b^i u(t), \tag{3}$$

where $i = 1, 2, \ldots, n$; b^i are all constant numbers.

Let us also write down system (3) in vectorial form:

$$\dot{x} = f(x) + bu, \tag{4}$$

where $\dot{x} = (x^1, x^2, \ldots, x^n)$, $x = (x^1, \ldots, x^n)$, $b = (b^1, \ldots, b^n)$.

The optimal control problem can be formulated in the following manner:

let us consider two points γ_0 and γ_1 in the phase space

x of system (3) $(x^1, \ldots, x^n)$; it is required to determine a permissible control $u = u(t)$ such that the describing point $x(x^1, \ldots, x^n)$ of the system (3) will move from the initial point γ_0 to the final point γ_1 in the shortest time.

Without placing any restriction on the generality of the problem, it is possible to assume that the system was at point γ_0 at time $t = 0$, reaching point γ_1 at time $t = T$, where T is the duration of the optimal process.

2. Basic Relations

It must be noted that the Maximum Principle is merely a necessary condition for the existence of an optimal process. Let us assume that there exists an optimal control $\overline{u} = \overline{u}(t)$ and a corresponding optimal locus $\overline{x} = \overline{x}(t)$ for $0 \leq t \leq T$ obtained from system (3) for $u = u(t)$ and connecting points γ_0 and γ_1 with an optimal transition time equal to T. Then this control $\overline{u} = \overline{u}(t)$ and the corresponding locus $\overline{x} = \overline{x}(t) = [\overline{x}^1(t), \ldots, \overline{x}^n(t)]$ satisfy system (3) for $0 \leq 1 \leq T$, i.e.,

$$\dot{\overline{x}}^i = f_i(\overline{x}^1, \ldots, \overline{x}^n) + b^i \overline{u}(t), \quad i = 1, 2, \ldots, n;$$

$$\overline{x}(0) = \gamma_0, \quad \overline{x}(T) = \gamma_1. \tag{5}$$

Let us denote by $\delta u = \delta u(t)$ the various permissible variations of the optimal control $\overline{u} = \overline{u}(t)$, i.e., $\overline{u} + \delta u(t)$ is a piecewise continuous function of time t and $|\overline{u}(t) + \delta u(t)| \leq 1$ for $0 \leq t \leq T$. Further, let us denote by $\delta x = \delta x(t)$ the variations of the optimal locus $\overline{x} = \overline{x}(t)$ corresponding to the variations $\delta u = \delta u(t)$ of the optimal control $\overline{u} = \overline{u}(t)$; moreover, $\delta x(0) = 0$ and $\delta x(T) = 0$, since the variational loci must pass through the points γ_0 and γ_1 lying on the optimal locus.

Let us now set $u = u(t) = \overline{u}(t) + \delta u(t) = \overline{u} + \delta u$ and $x = x(t) = \overline{x}(t) + \delta x(t) = \overline{x} + \delta x$ and make the substitutions into system (3).

Separating from the right side of each equation in system (3) the first-degree terms in $\delta x^i = \delta x_i(t)$ and making use of equations (5), we obtain

$$\delta \dot{x}^i = \sum_{\alpha=1}^{n} \frac{\partial f_i}{\partial x^\alpha} \delta x^\alpha + \ldots + b^i \delta u,$$

$$i = 1, \ldots, n. \tag{6}$$

System (6) is a so-called equation in terms of variations; terms of order higher than one are designated by three dots.

System (6) can be briefly written in the vector-matrix form

$$\delta \dot{x} = A\,\delta x + \ldots + b\,\delta u, \tag{7}$$

where $A = \left\| \frac{\partial f^i}{\partial x^\alpha} \right\|$ is an n-th order square matrix.

Examining first the system of equations of the first approximation obtained from equations (6) and (7) by neglecting higher-order terms in δx^i, we have:

$$\dot{\xi}^i = \sum_{\alpha=1}^{n} \frac{\partial f^i}{\partial x^\alpha} \xi^\alpha + b^i \delta u, \quad i = 1, \ldots, n. \tag{8}$$

It is possible to prove that ξ^i is the first approximation to δx^i, i.e.,

$$\frac{|\xi(t) - \delta x(t)|}{\eta} \to 0 \text{ for } |\delta u(t)| = \eta \to 0,$$

$$0 \leq t \leq T.$$

In vector-matrix form, system (8) assumes the form

$$\dot{\xi} = A\xi + b\delta u;$$

$$\dot{\xi} = (\dot{\xi}^1, \dot{\xi}^2, \ldots, \dot{\xi}^n), \quad \xi = (\xi^1, \xi^2, \ldots, \xi^n), \quad A = \left\| \frac{\partial f^i}{\partial x^\alpha} \right\|.$$

Next, let us denote by $G(t_1)$, $0 \leq t_1 \leq T$, the set of points of the form $\overline{x}(t_1) + \xi(t_1)$; these are the same points which may be reached at time t_1 moving with the aid of an arbitrary but permissible control $u(t) = \overline{u}(t) + \delta u(t)$ in accordance with equations (8) and (9) with initial conditions. We shall prove that for any t_1, $0 \leq t_1 \leq T$, the domain $G(t_1)$ is a convex set. Let us take two arbitrary points belonging to domain $G(t_1)$,

$$x_1(t_1) = \overline{x}(t_1) + \xi_1(t_1) \tag{9}$$

and

$$x_2(t_1) = \overline{x}(t_1) + \xi_2(t_1),$$

obtained from the permissible controls $\overline{u}(t) + \delta u_1(t)$ and $\overline{u}(t) + \delta u_2(t)$, $0 \leq t \leq t_1$.

Let us show that point $x_0(t_1) = \lambda x_1(t_1) + \mu x_2(t)$ for $\lambda \geq 0, \mu \geq 0, \lambda + \mu = 1$, lying on the segment joining points $x_1(t_1)$ and $x_2(t_1)$ is also included in domain $G(t_1)$, i.e., that $x_0(t_1)$ belongs to $G(t_1)$. And we actually obtain that

$$x_0(t_1) = \lambda x_1(t_1) + \mu x_2(t_1) =$$
$$= \lambda[\bar{x}(t_1) + \xi_1(t_1)] + \mu[x(t_1) + \xi_2(t_1)] =$$
$$= \bar{x}(t_1) + \lambda \xi_1(t_1) + \mu \xi_2(t_1).$$

It is hence apparent that the point $x_0(t_1)$ is obtained with the aid of the control $\bar{u}(t) + \lambda \delta u_1(t) + \mu \delta u_2(t)$, $0 \leq t \leq t_1$. It remains to show that this equation is permissible.

We have that

$$|\bar{u} + \lambda \delta u_1 + \mu \delta u_2| = |\lambda (\bar{u} + \delta u_1) + \mu(\bar{u} + \delta u_2)| \leq$$
$$\leq |\lambda(\bar{u} + \delta u_1)| + |\mu (\bar{u} + \delta u_2)| \leq \lambda + \mu = 1,$$

since $|\bar{u} + \delta u_1| \leq 1$ and $|\bar{u} + \delta u_2| \leq 1$.

It follows from this that $x_0(t_1)$ belongs to $G(t_1)$, i.e., domain $G(t_1)$ includes, along with some two points, the entire segment connecting these points; this means that $G(t_1)$ is a convex set.

Let us show now that the point $\bar{x}(t_1)$ of the optimal locus for $0 \leq t_1 \leq T$ lies exactly on the boundary of the convex domain $G(t_1)$. And actually, the point $\bar{x}(t_1)$ cannot lie on the exterior of the domain $G(t_1)$ by virtue of the definition of this domain (see page 4).

It remains for us to prove that the point $\bar{x}(t_1)$ cannot lie strictly on the interior of domain $G(t_1)$. The assumption that the point $\bar{x}(t_1)$ lies strictly within the domain $G(t_1)$ leads, as is proved, to a contradiction of the fact which we assumed at the very outset, that the locus $\bar{x} = \bar{x}(t)$, $0 \leq t \leq T$ is optimal, since it is then possible to reach point $\bar{x}(t_1)$ in a time shorter than t_1.

Since the set $G(t)$, $0 \leq t \leq T$, is convex, it is possible to draw a support hyperplane through point $\bar{x}(t)$, $0 \leq t \leq T$, such that $G(t)$ will lie on one side of P. At point $\bar{x}(t)$ let us take the vector

$$p(t) = (p^1(t), p^2(t), \ldots, p^n(t)),$$

orthogonal to plane P and directed out of domain $G(t)$.

Since the vector $\xi(t)$ likewise directed out of point $\bar{x}(t)$ and has its end point in $G(t)$, as a result of the convexity of $G(t)$, we obtain the scalar product

$$p(t)\cdot \xi(t) \leq 0 \quad \text{for} \quad 0 \leq t \leq T, \tag{10}$$

where the vector $\xi(t) = \xi^1(t), \xi^2(t), \ldots, \xi^n(t)$ is a solution of the system (9) for any permissible control $\bar{u} + \delta u$.

The general solution of the non-homogeneous system of linear equations (9) with the initial conditions $\xi^i(0)=0$, $i = 1,\ldots,n$ has the form

$$\xi^i(t) = \sum_{\alpha=1}^{n} \varphi_\alpha^i(t) \int_0^t \sum_{\beta=1}^{n} \psi_\beta^\alpha(\tau) b^\beta \delta u(\tau) d\tau \tag{11}$$

where

$$\varphi_j(t) = (\varphi_j^1(t), \varphi_j^2(t), \ldots, \varphi_j^n(t)), \; j = 1,2,\ldots,n$$

is a fundamental system of solutions to the homogeneous linear system corresponding to system (9), and $\|\psi_\beta^\alpha(t)\|$ is the inverse of matrix $\|\varphi_\alpha^i(t)\|$.

As a result of the necessary condition (10) for the optimal solution and equation (11), we obtain:

$$p(t)\xi(t) = \sum_{i=1}^{n} p^i(t)\xi^i(t) =$$

$$= \sum_{i=1}^{n} p^i(t) \sum_{\alpha=1}^{n} \varphi_\alpha^i(t) \int_0^t \sum_{\beta=1}^{n} \psi_\beta^\alpha(\tau) b^\beta \delta u(\tau) d\tau =$$

$$= \int_0^t \sum_{\beta=1}^{n} \psi_\beta(\tau) b^\beta \delta u(\tau) d\tau = \int_0^t \psi(\tau) b \delta u(\tau) d\tau \leq 0 \tag{12}$$

where $\psi(t) = (\psi_1(t),\ldots,\psi_n(t)), \psi_\beta(\tau) =$

$$= \sum_{i=1}^{n} \sum_{\alpha=1}^{n} p^i(t)\varphi_\alpha^i(t)\psi_\beta^\alpha(\tau),$$

$$\beta = 1,2,\ldots,n.$$

Hence, the optimum solution requires that the following condition be satisfied:

$$\int_0^t \psi(\tau) b\, \delta u(\tau) d\tau \leq 0. \tag{13}$$

From inequality (13) it follows that the optimum control

$$u(t) = \operatorname{sign} \psi(t) b, \quad 0 \leq t \leq T. \tag{14}$$

Actually, since the function sign x is a piecewise continuous function of its argument and does not exceed unity in modular value, the control (14) is indeed permissible. Now let us consider the opposite case, i.e., that $\psi(t)b > 0$ and $u(t) < 1$ for $0 \leq t \leq T$. In this case we assume

$$u(t) = \begin{cases} \alpha = \text{const} > 0 \text{ for } (t)b < 0, \\ 0 \text{ in all other cases.} \end{cases}$$

Then, as a consequence of equation (13), we have that

$$\int_0^t \psi(t) b\, \delta u(\tau) d\tau = \int_0^t \psi(t) b \alpha d\tau > 0, \quad 0 \leq t \leq T.$$

Thus, we obtain a contradiction with the aid of condition (13); hence we conclude that for $\psi(t)b > 0$, $u(t) = 1$ in all cases.

In a completely analogous manner, it is possible to show that for $\psi(t)b < 0$ we obtain $u = -1$. Consequently, equation (14) has been proved.

3. The Determining System of Differential Equations for Vector $\psi(t)$

Let us derive the differential equations which determine the vector $\psi(t) = (\psi^1(t), \psi^2(t), \ldots, \psi^n(t))$, $0 \leq t \leq T$.

As a result of the fact that $\|\varphi_j^i(t)\|$ and $\|\psi_j^i(t)\|$ are mutually inverse matrices, we have that

$$\varphi_i(t) \psi^j(t) = \delta_i^j = \begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j. \end{cases}$$

Differentiating this equation with respect to t, we obtain

$$\dot{\varphi}_i(t)\psi^j(t) + \varphi_i(t)\dot{\psi}^j(t) = 0. \tag{15}$$

Since $\varphi_i(t)$ satisfies the equation $\dot{\varphi}_i(t) = A\varphi_i(t)$, expression (15) may be rewritten as

$$A\varphi_i\psi^j + \varphi_i\dot{\psi}^j = 0,$$

$$\varphi_i A'\psi^j + \varphi_i\dot{\psi}^j = 0,$$

$$\varphi_i(A'\psi^j + \dot{\psi}^j) = 0,$$

where A' is the conjugate of matrix A.

The last equation is satisfied for any $i = 1,2,\ldots,n$ and $\varphi_1, \varphi_2, \ldots, \varphi$ are linearly independent vectors for any $0 \leq t \leq T$, as a fundamental system of solutions. Thus, we obtain that

$$\dot{\psi}^j = -A'\psi^j; \quad j = 1,2,\ldots,n. \tag{16}$$

Since the vector $\psi(t)$ is a linear combination of vectors $\psi_j(t)$, $j = 1,2,\ldots,n$, $\psi(t)$ will also satisfy equation (16), i.e.,

$$\dot{\psi}(t) = -A'\psi(t). \tag{17}$$

Further, let us write down the equation $\psi(t)$ for the case of the initial system (2):

$$\left.\begin{aligned} \dot{\psi}^1 &= -\frac{\partial f}{\partial x^1}\psi^n; \\ \dot{\psi}^2 &= -\psi^1 - \frac{\partial f}{\partial x^2}\psi^n; \\ &\cdots\cdots\cdots\cdots\cdots \\ \dot{\psi}^n &= -\psi^{n-1} - \frac{\partial f}{\partial x^n}\psi^n. \end{aligned}\right\} \tag{18}$$

As a result of expression (14), the optimal control in this case will be $u = \operatorname{sign}\psi_n(t)$, since it is necessary to set $b = (0,0,\ldots,0,1)$. Let us now show that the expression

$\psi_n(t) \not\equiv 0$ on any interval $(t_1, t_2) \subset [0,T]$ if $\psi_n(0) \neq 0$.

Let us make the contrary assumption that for $t \in (t_1,t_2) \subset [0,T]\ \psi_n(t) \equiv 0$. Then as a result of the last equation of (18), we obtain that

$$\psi_{n-1} = -\dot{\psi}_n - \frac{\partial f}{\partial x^n}\psi_n \equiv 0 \text{ on } (t_1,t_2) \subset [0,T].$$

From the second-to-last equation of system (18) we obtain that

$$\psi_{n-2} = -\dot{\psi}_{n-1} - \frac{\partial f}{\partial x^{n-1}}\psi_n \equiv 0 \text{ on } (t_1,t_2) \subset [0,T],$$

since it has already been proved that $\psi_{n-1} \equiv \dot{\psi}_{n-1} \equiv 0$, etc. up to the first equation of system (18).

Thus, we have that

$$\psi_1(t) \equiv \psi_2(t) \equiv \dots \equiv \psi_n(t) \equiv 0 \text{ on } (t_1,t_2) \subset [0,T].$$

But this solution is obtained only in the case $\psi(0) = 0$, due to the uniqueness of the solution. We now have a contradiction to the statement that $\psi(0) \neq 0$. This proves that the control $u(t) = \operatorname{sign}\psi_n(t)$ is determined almost throughout $[0,T]$.

On the basis of these results, it is possible to formulate the optimum condition in the form of L.S. Pontryagin's Maximum Principle.

Let us consider the scalar product

$$\psi(t)\dot{x}(t) = \sum_{\alpha=1}^{n} \psi_\alpha(t)\dot{x}^\alpha(t),$$

which on the basis of system (3) is equal to

$$\psi(t)\dot{x}(t) = \sum_{\alpha=1}^{n} \psi_\alpha(t)\dot{x}_\alpha(t) = \sum_{\alpha=1}^{n} \left[\psi_\alpha(t) f_\alpha(x(t)) + \psi_\alpha(t) b^\alpha u(t)\right]. \tag{19}$$

For fixed vectors $x(t)$ and $\psi(t)$ and a varying parameter $u(t)$ within the limited range $|u(t)| \leq 1$, the last expression (19) reaches a maximum according to equation (14).

<u>The Maximum Principle.</u> In order for u(t) to be an optimal control, it is necessary that the function

$$H(x,\psi_1,u) = \psi(t)\dot{x}(t) = \sum_{\alpha=1}^{n}[\psi_\alpha(t)f_\alpha(x(t)) + \psi_\alpha(t)b^\alpha u(t)]$$

reach a maximum along u(t) as this piecewise continuous function varies within the interval $[-1,1]$. The 2n-dimensional (x,ψ) vector, moreover, is the solution to the following Hamiltonian system:

$$\dot{x}^1 = \frac{\partial H}{\partial \psi_1(t)};$$
$$\dot{\psi}^1 = -\frac{\partial H}{\partial x^1}.$$

For the sake of illustration, let us examine the derivation of an optimal control for the simplest second-order linear system. The control u(t), limited in its modulus to $|u(t)| \leq 1$, is fed into the input of an aperiodic component with time constant T and gain k_2. The signals at the output of the aperiodic component x_2 are fed to the input of an integrator of gain k_1 and output x_1. The equations describing such a dynamic system will have the form

$$\left.\begin{aligned} \dot{x}_1 &= k_2x_2; \\ T\dot{x}_2 &= -x_2 + k_1u, \quad |u(t)| \leq 1. \end{aligned}\right\} \quad (20)$$

Substituting in:

$$z_1 = \frac{T}{k_1k_2}x_1; \quad z_2 = \frac{T}{k_1}x_2, \text{ equations (20)}$$

can be rewritten

$$\left.\begin{aligned} \dot{z}_1 &= z_2; \\ \dot{z}_2 &= -\frac{1}{T}z_2 + u, \quad |u(t)| \leq 1. \end{aligned}\right\} \quad (21)$$

Matrices A and A' in this case have the form

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{T} \end{pmatrix}; \quad A' = \begin{pmatrix} 0 & 0 \\ -1 & \frac{1}{T} \end{pmatrix};$$

consequently, for $\psi(t)$, the system takes the form:

$$\left.\begin{aligned} \dot{\psi}(t) &= 0; \\ \dot{\psi}_2(t) &= -\psi_1(t) + \frac{1}{T}\psi^{2}, \end{aligned}\right\} \qquad (22)$$

whence,

$$\left.\begin{aligned} \psi^1(t) &= C; \\ \dot{\psi}^2(t) &= \frac{1}{T}\psi_2(t) - C. \end{aligned}\right\} \qquad (23)$$

Further, we find that $\psi_2(t) = De^{t/T} + D_1$, where D and D_1 are arbitrary constants.
Thus, the optimal control is of the form $u(t) = \text{sign}\ (De^{t/T} + D_1)$, the constants D and D_1 being determined by the condition that the locus $x(t)$ reach the required point at the moment of process termination.

It is evident that the function $\psi_2(t) = De^{t/T}$ changes sign not more than once -- i.e., the optimal process must consist of two time intervals, in each of which $u(t)$ takes one of its limiting values.

References

1. V.G. Boltyanskiy, R.V. Gamkrelidze, L.S. Pontryagin, "On Optimal Process Theory", DAN SSSR (Doklady Akademii Nauk SSSR -- Reports of the USSR Academy of Sciences), 110, 1956, No 1, Mathematics Section.
2. R.V. Gamkrelidze, "Optimal Process Theory in Linear Systems", Uspekhi Matematicheskikh Nauk (Advancements in the Mathematical Sciences), 1957, 12, issue 5 (77).
3. V.G. Boltyanskiy, "The Maximum Principle in Optimal Process Theory", DAN SSSR , 1958, 119, No 6, Mathematics Section.

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10,106 - Ed.

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